Walking patterns and folding paper

How does a "Wiskunde B"-day looks like?

In Belgium, the Netherlands, Germany, Slovakia we organize a day around IBME with pupils 17-18y. Every year a small committee, works out a subject.

Participating pupils work in groups of 4. They get time to work on the assignment from 9 to 16h. At the end of the day, they hand in a report with their story. The readability and accuracy of this report is of great importance for the rankings later on.

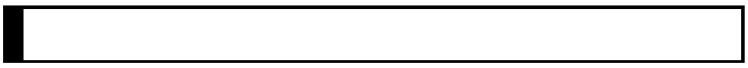
There are always two parts. In part A (exploration), pupils are given some suggestion and experience on the topic of that year that will be needed in order to get their research going. In part B (own research) they can choose from some research suggestions.

Assignments of the last years, can be found on http://www.fi.uu.nl/wisbdag/, and then look at "Opdrachten" in the left menu. Also the applets can be found there. The assignments in English start from 2006.

We are going to do a little part of the 2012 assignment (the original had 9 exploratory questions and 10 research questions). Pupils had also an applet, to help them in their research and to check their conjectures. Some questions are rather in open form, as you will see in part B. Have fun!

Part A: Exploration

You need strips of paper. The size does not matter. At the end of a strip you put a small black mark on one side. That is the starting point of the strip.



Place the strip on its side on the table in front of you *with the black mark facing you* as drawn above.
Now you will fold the strip a few times, without unfolding it in-between.
You always fold the right-hand side of the (folded) strip towards the left-hand side.

After folding like this a few times it looks complicated, but you can always find the starting point again because of the black mark.

There are always two ways to fold the strip:

- Folding away from you, so to the left. The right-hand side will end up *behind* the left-hand side. We indicate this with the letter l.

- Folding towards you, so to the right. The right-hand side will end up *in front* of the left-hand side. We indicate this with the letter r.

The folded strip of paper is folded again, and this is repeated a number of times. Each time it can be l or r.

A **folding recipe** is a series of (small) letters l and r. These are used to prescribe the folding actions. You follow the recipe from left to right. An example will clarify this.

There are three folding steps in the folding recipe *l r r*:

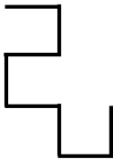
Fold 1: *l*, so to the left	Fold 2: *r*, so to the right	Fold 3: *r*, so to the right
		

After performing the folding recipe you do the following:

- Unfold the strip. Take care to have 90 degree angles on all folding lines and to keep the connecting pieces straight.
- Put the unfolded strip on its side in front of you on the table, with the black mark facing you on the left.

You can see the result on the photo on the right.

From the top you can see the shape that is drawn on the right. We call this type of figure a **walking pattern**. Mainly we draw the walking pattern so that the **start is from the black mark facing horizontally to the right**.

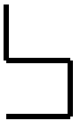


The walking pattern can also be described as a route from the starting point to the end of the strip past all bends. Imagine that you are following the walking pattern from start to finish. Each time you walk along a stretch of straight road, followed by a 90 degree bend either to the left or to the right. After the next straight stretch of road, there will be another bend to the left (L) or to the right (R). Only the bends matter; the stretches in between are all the same length.

The route for the walking pattern in the example is *RRLLRLL.*
For simplicity's sake, we will also call the series of letters *L* and *R* the **walking pattern**, just like the figure itself.
So there are two important notions in folding and unfolding the strips:

- the **folding recipe**. This indicates how to fold, using a series of (small) letters *l* and *r*.
- the **walking pattern**. This is the result after following a folding recipe, described in a figure or a series of (capital) letters *L* and *R*.

Introductory research
Below are the four folding recipes that fold twice. The illustrations all start horizontally to the right, but that is just to show where and how to start.

Folding recipe	Walking pattern
r r	RRL 
r l	LRR 
l r	RLL 
l l	LLR 

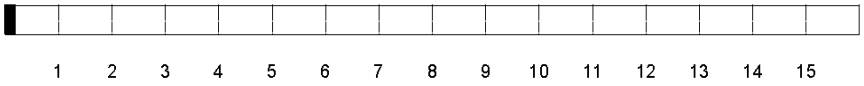
Of course you check the four possible patterns by performing the folding recipes and looking carefully at the results and how they arose.
To further familiarize yourself with folding recipes and their walking patterns, you will first perform all folding recipes for three folds.

Exploratory question 1.

a. There are eight possibilities for a three-fold folding recipe. What are they?
b. Make an overview for the three-fold recipes, like the one above for two-fold recipes: both the folding recipes and their walking patterns.
c. You won't find the zigzag pattern below as a possible three-fold pattern. You could have known in advance. How?

The order of the bends

Four folds will give you 16 straight pieces and 15 bends after unfolding.
Below, the dotted folding lines have been numbered left to right in the completely unfolded strip. These are the positions on the strip where bends will appear in the walking pattern.

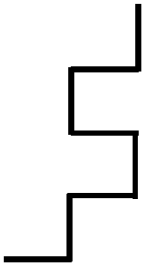


Exploratory question 2.

a. How many straight pieces do you get when you fold the strip n times? How many bends are there in that case?
b. In which position (which number in the drawing above) will you find the folding line for the initial fold? And which positions (numbers) will indicate the second, third and fourth fold?
   There is an *L* or an *R* in each position, depending on the folding recipe. Someone has made a walking pattern with an *R* in the positions 2, 7, 8 and 12. That is enough information to retrieve the whole walking pattern!
c. Determine the whole walking pattern (give the series of *R*s and *L*s).
d. What was the folding recipe?

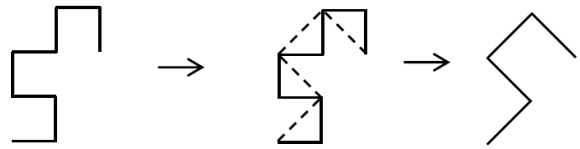
Exploratory question 3.

a. Why can the figure on the right not occur as a walking pattern?
b. Why can RLLRRRLLRLLRRRL not occur as a walking pattern?



The derivative of a walking pattern

Starting with a walking pattern, one can get a pattern by connecting bends, but skipping one bend each time. We call this the *derivative* of a walking pattern. Look at the example (dotted line is the derivative).

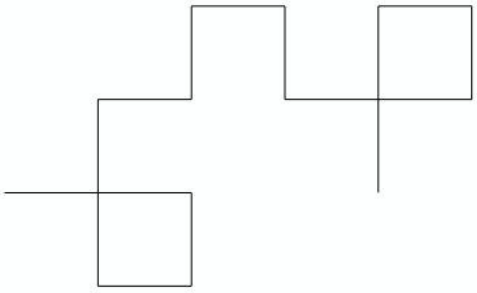


Exploratory question 4.

a. The left walking pattern comes from the folding recipe *r l l* (check with your list in question1). Proof that the derivative (on the right) is also a walking pattern. Try to find the folding recipe.
b. Can you explain this in general, i.e. the derivative of a walking pattern is always again a walking pattern (with a folding recipe that is known)?

Exploratory question 5.

a. Look at the walking pattern from the folding recipe *r l l r*. What will change (in the walking pattern) if the first *r* in the folding recipe is replaced by an *l* ?
b. What will change if we replace the final *r* with an *l* ?



Relation between walking pattern and the folding recipe

Below you find a schematic representation how the final walking pattern emerges from the folding recipe r l r l. The position is the number of the bend in the final walking pattern. From this you should try to understand the relation between the pattern and the recipe.

after fold 4	*L*	*R*	*R*	*L*	*L*	*L*	*R*	*R*	*L*	*R*	*R*	*R*	*L*	*L*	*R*
after fold 3		*R*		*L*		*L*		*R*		*R*		*R*		*L*	
after fold 2				*L*				*R*				*R*			
after fold 1								*R*							
position	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Exploratory question 6.

a. You get three bends in a walking pattern of a 4-folded strip. On position 3-4, you get LR, on position 6 you get R. Do you know what the bend on position 10, 12 and 13 will be?
b. Can you deduce the folding recipe from the three given bends? Explain in detail.

Part B: Your own research

Below, some research suggestions are given. Select among the suggestions or explore your own research questions. Describe clearly what you investigated and what your findings were.

Research question 1.
For every walking pattern, the first and the last straight piece are perpendicular to each other. Try to find a solid reasoning that shows this is always true.

Research question 2.
Select a fixed number n (= 1, 2, 3, 4, 5, ...). For all walking patterns that you can make using the chosen n folds, the distance from start to finish is the same. Investigate why this is so.
Note: that distance of course depends on the length of the strip you start with. It makes sense to start with a strip of length 2^n, so that all straight pieces will have length 1.

Research question 3.
Select a fixed number n (= 1, 2, 3, 4, 5, ...).
Mark the starting point of a walking pattern in a fixed place on a sheet of grid paper.
- Where are the possible end points of a walking pattern after n folds now?
- On the way from the starting point to the end point there may be points that are further away from the starting point than the end point is. Investigate what distances are possible. Can you express the maximum distance from the starting point to any point in the walking pattern in n?

Research question 4.
- Can you prove that RRRR (or LLLL) is not possible in a walking pattern?
- When do you run into points of contact that only occur later in the pattern?

Part B: Your own research

Below, some research suggestions are given. Select among the suggestions or explore your own research questions. Describe clearly what you investigated and what your findings were.

What can you tell about:

- The orientation of the first and the last straight line in a walking pattern?
- Fix the number of foldings (n) and suppose a fixed length of the paper strip: what can you say about the distance of start- and endpoint of the walking pattern?
- Is the endpoint always at the maximal distance from the starting point, or lie other point further away?
- Are there possible problems in unfolding a folded strip of paper? E.g. LLLL or RRRR gets you into trouble!
- There seems to occur points of contact (points that occur later again in the walking pattern): are there unfolding problems possible there? When do this contact points occur?
- What kind of symmetries do occur in the walking patterns?