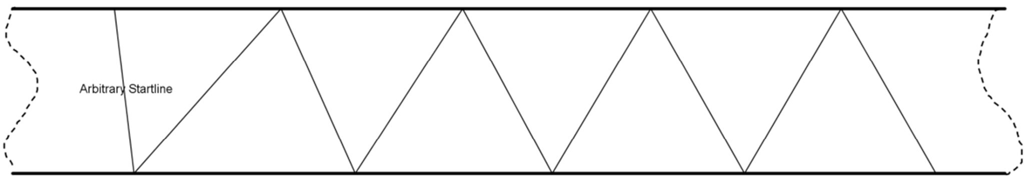
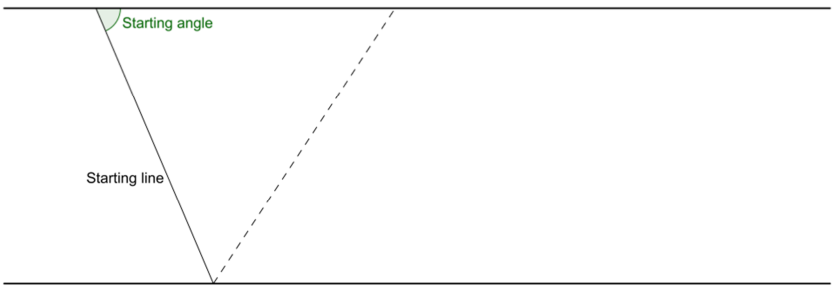
Assignment 1

- ✓ Start with an arbitrary folding line on your paper roll.
- ✓ Do action 'Folding Up' (U) to create a new folding line
- ✓ Do action 'Folding down' (D) to create a new folding line
- ✓ Repeat this (doing U, then D) until you have 14 'triangles' on your paper strip



- ✓ Can you see that the triangle seems to become more equilateral as we proceed folding? We want to explain this: calculate the angles on your paper strip if your first angle was exactly 70° (see picture below)? After how many folding actions is the angle close to 60° ('close' meaning: a difference less than 1°)?



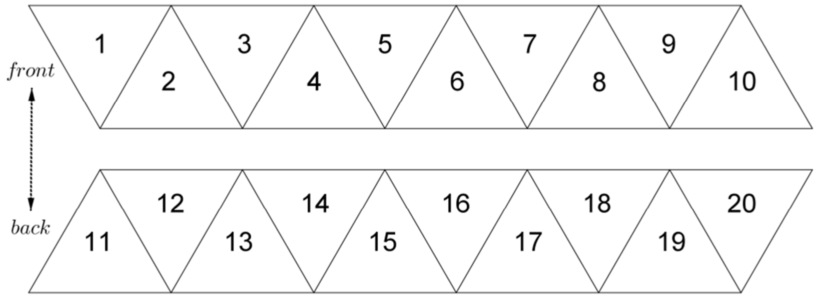
- ✓ Is it clear than any starting angle will do? If you are not convinced yet, calculate the angles when your arbitrary starting angle was 50°. What if the starting angle was 60°?
- ✓ Now, cut of the strip with folding lines in order to end up with a strip of 14 triangles.
- ✓ Cut off the first 4 triangles (make sure that you cut off the triangles that were folded first). Now you are left with a strip of exactly 10 triangles, which are all more or less equilateral.
- ✓ Make a flexagon (see instructions and learn to flex.
- ✓ Color every face in another color and flex again. What do you notice?
- ✓ At last, try to put on a smiling face on the front and an angry face on the back of your flexagon. Do this first with a very gentle pencil, because you will have to make the face nicer afterwards. What do you notice when you are flexing.
- ✓ Congratulations. Have fun with the flexagon. Can you explain why a sad face is turning into a smiling face (and vice versa) after flexing? More concrete question: can you describe the effect of flexing on your polygon in terms of geometry (e.g. rotations)?

Folding a flexagon

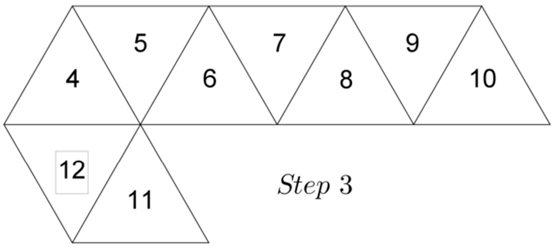
Follow each step of the procedure to fold a strip of 10 equilateral triangles into a flexagon with 3 faces.

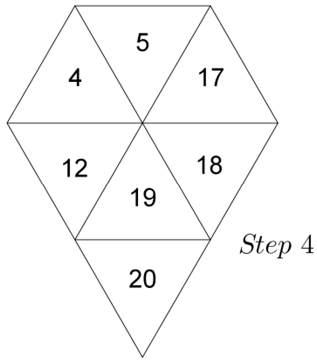
1. Start with a strip of 10 equilateral triangles. Label each triangle as in the picture below. Make sure of the correct orientation: triangle 1 has its top downwards! Important: triangle 11 is

the backside of triangle 1,…

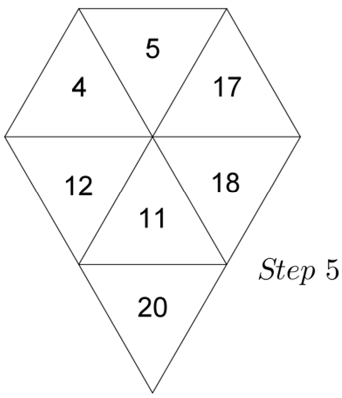


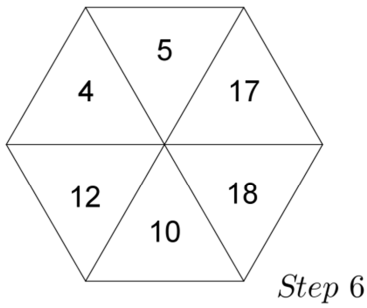
2. Make sure all folding lines are folded in both directions. We need all lines to be very flexible.
3. Start with triangles 1-10, triangle 1 top downwards. Fold triangles 1-3 underneath triangles 4-10.





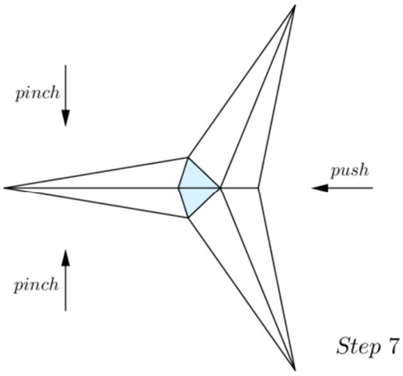
4. Fold triangles 7-10 on top of triangle 6.
5. Attention! Place now triangle 11 <u>on top</u> of triangle 19.

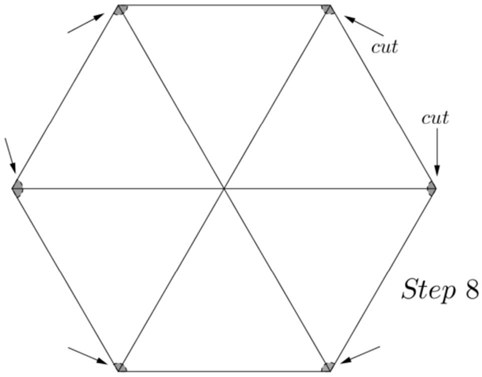




6. Final step is to fold triangle 20 on triangle 11. You get a regular hexagon as in the picture. If you glue triangle 11 and 20 together, the flexagon is ready.

7. How to flex? Choose two adjacent triangles and pinch them together. Push the middle of the other 4 triangles towards the center (you get a 'windmill' like in the picture below). If you choose two good triangles to pinch, the center will open a little and the triangles that were hidden will appear.
   If the center does not open: you'll have to pinch another couple of triangles (it is wise to take one of the triangles you tried first, but pinch it together with his other adjacent triangle).

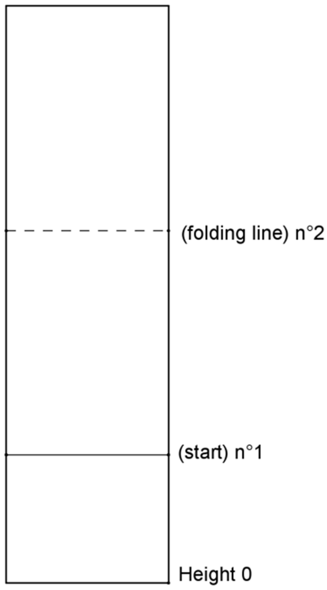




8. When flexing is not going very smoothly, you probably did not fold precisely enough. One way to solve this, is to cut in every edge a little bit as in the picture. Then there will be more space to flex.

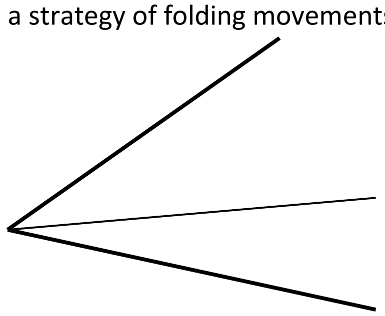
Assignment 3

You get a strip of paper and we want to divide this into 1/5 just by folding, not by measuring anything.



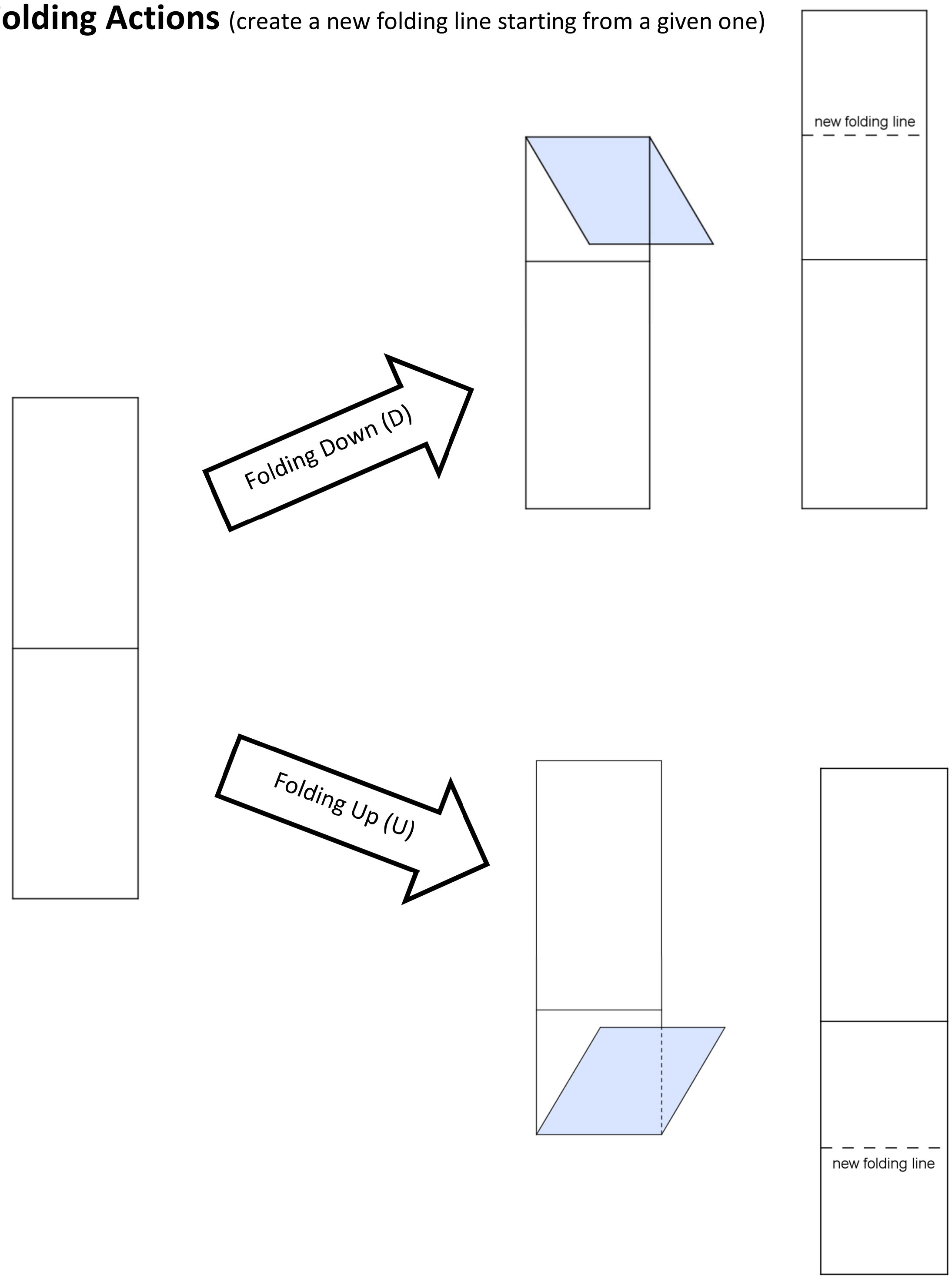
- ✓ Just lay the strip in front of you, where the height is the long side. Start with a wild guess where 1/5 is (starting with measuring down to top) and fold a horizontal line. Put a number 1 close to it.
- ✓ Now we do the action D (folding downwards): fold the top edge of the paper strip on top of line number 1. This creates a new folding line. Put a number 2 close to it.
- ✓ Do again action D: fold the top edge on top of line number 2 and create by doing so, line number 3.
- ✓ Next we do action U (folding upwards): fold the bottom edge of the paper strip on top of line 3. By doing so, you create line 4.
- ✓ Do again action U to create line 5.
- ✓ Line 5 will be on 1/5 of the height of the total strip. If you have a ruler, you can measure that.
- ✓ Take another paper strip, and now start with a very bad guess to start with (close to or even above half of the strip). This is your new line 1. Now execute as above the different folding actions: D, D, U and U. Is line 5 again at $1/5^{th}$ of the total strip?
- ✓ We want to explain what we are seeing. Suppose that your first guess was correct: so at $1/5^{th}$ of the strip. Now calculate the heights of line 2-5 (suppose that the strip has height 1 and the bottom edge is at height 0).
- ✓ Of course, the fact that the first guess is spot on, is very unlikely to happen. Suppose that your first guess (and therefore the height of line 1) was at height 1/5 + e. The letter e stands for 'error' and can be both positive and negative. Now calculate again the heights of line 2-5 in this case.
- ✓ Is line 3 also a good line to be at $1/5^{th}$, but now from top to bottom? Explain.
- ✓ Formulate a correct conclusion to what you have seen.
- ✓ What would you do if you were asked to fold a line at height $2/5^{th}$ from the bottom edge, starting with an arbitrary line?

- Extra 1. Suppose you have an arbitrary angle you want to divide in 5 (see picture). Can you do this by folding again? Start with a guess of $1/5^{th}$ of the angle as your first folding line. Express a strategy of folding movements in order to end up with $1/5^{th}$ of the given angle.



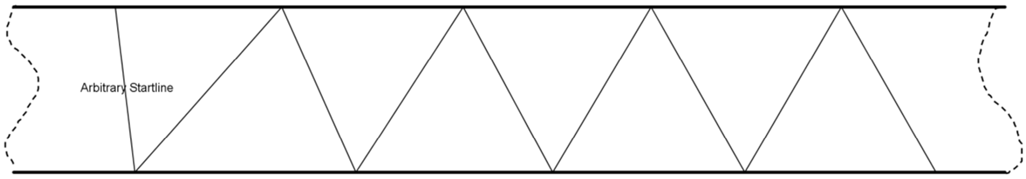
- Extra2. Start with a new strip of paper and make a starting line n° 1 (again around 1/5th above the bottom edge).
- Execute actions D, U and U to create lines 2-4. In what part can you divide a strip by this folding procedure? In other words, what is the height of line 4?

Folding Actions (create a new folding line starting from a given one)

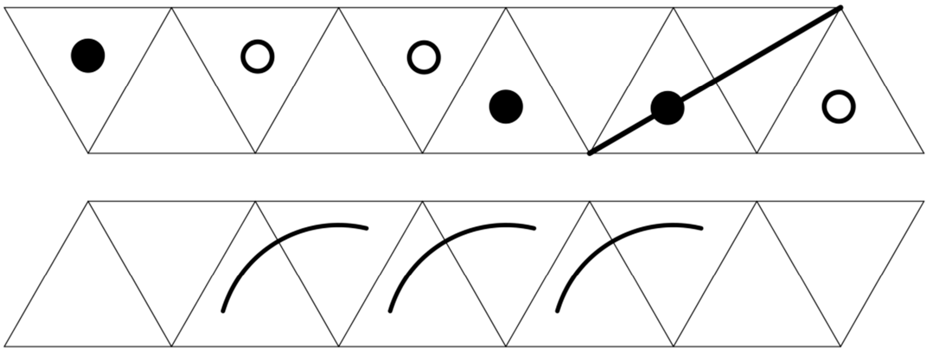


Assignment 2

- ✓ Start with an arbitrary folding line on your paper roll.
- ✓ Do action 'Folding Up' (U) to create a new folding line
- ✓ Do action 'Folding down' (D) to create a new folding line



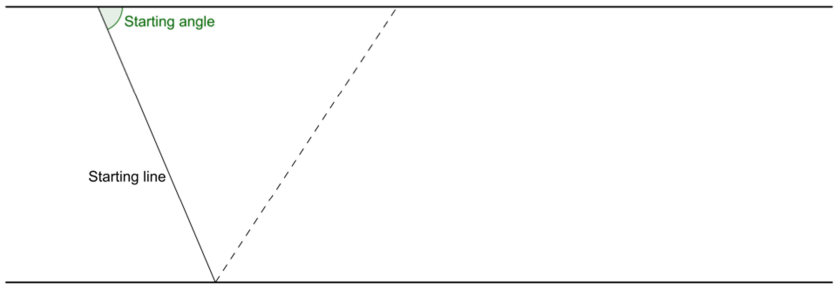
- ✓ Repeat this (doing U, then D) until you have 14 'triangles' on your paper strip
- ✓ Can you see that the triangle seems to become more equilateral as we proceed folding?
  Now, cut of the strip with the first and the last folding lines. This gives you a strip of 14 triangles.
- ✓ Cut off the first 4 triangles (make sure that you cut off the triangles that were folded first). So you are left with a strip of exactly 10 triangles, which are all more or less equilateral.
- ✓ Carefully draw the following figures 0with gentle pencil on the strip as shown below.
  !!! Make sure that the first triangle of the strip above is pointing downwards (as in triangle 1 of the folding instructions)
  !!! Make sure that the left triangle in the second row is the back of the left triangle of the first row (triangle 11 in the folding instructions).



- ✓ Make a flexagon following the instructions (see other paper) and learn to flex. You can color each face in a different color.
- ✓ Start with a configuration with a smile on the front side, and start to flex. After three flexes, you seem back to the original face you started with… nut that is only partial true.
  Now look very careful and you will notice that after three times flexing the original face is again visible but is a little bit rotated: in order to see this you will have to look very good to the edges of the hexagon while flexing. Start to figure out what exactly has happened after three flexes.
- ✓ How many flexes do you need in order to get back to the original starting position (both face and orientation). Make a list of all different faces you saw in the front from start to the end.
- ✓ You will notice that you only have seen smiling faces. How do you get non-smiling faces (it is not that difficult)? We call this action 'flip'.

- ✓ Investigate the following question: does the order of your actions matter to the result? In other words: does “first flexing (a number of times) and then flipping” generates the same result as “first flipping and then flexing (the same number of times)”?
- ✓ Do you know any other figure that has the same characteristics? In more concrete words: flexing and flipping let the form of the figure the same, but not the points itself… do you know a (regular) figure that has the same sort of invariance actions?

- Extra. We want to explain the fact that triangles get more equilateral if you proceed folding: calculate the angles on your paper strip if your first angle was exactly 70°?

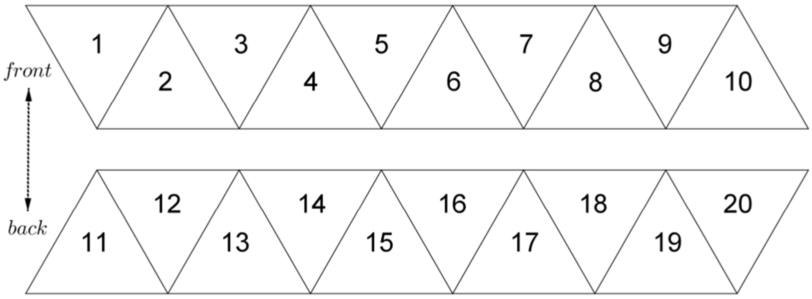


- Calculate the angles when your arbitrary starting angle was 50°. What if the starting angle was 60°?
- Can you deduce a waterproof argument that starting from an arbitrary angle, the triangles you become while doing D, and then U are converging to equilateral triangles?

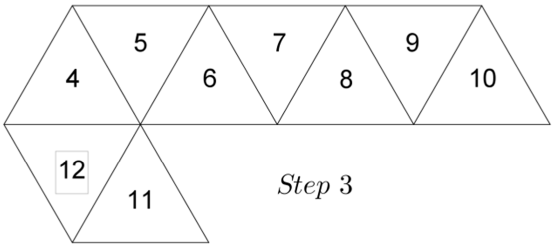
Folding a flexagon

Follow each step of the procedure to fold a strip of 10 equilateral triangles into a flexagon with 3 faces.

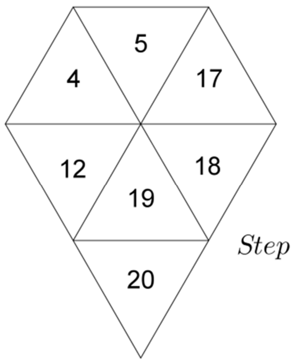
1. Start with a strip of 10 equilateral triangles. Label each triangle as in the picture below. Make sure of the correct orientation: triangle 1 has its top downwards! Important: triangle 11 is the backside of triangle 1,…



2. Make sure all folding lines are folded in both directions. We need all lines to be very flexible.
3. Start with triangles 1-10, triangle 1 top downwards. Fold triangles 1-3 underneath triangles 4-10.

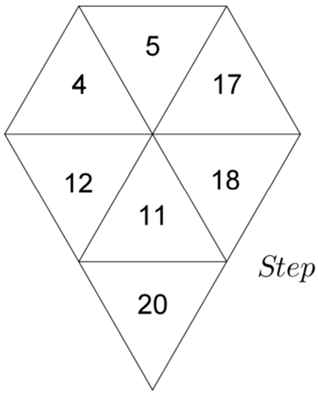


Step 3

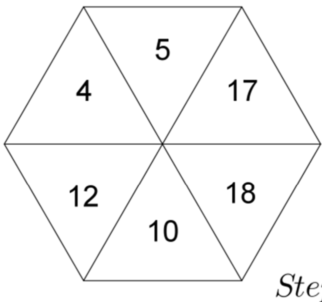


Step 4

4. Fold triangles 7-10 on top of triangle 6.
5. Attention! Place now triangle 11 <u>on top</u> of triangle 19.



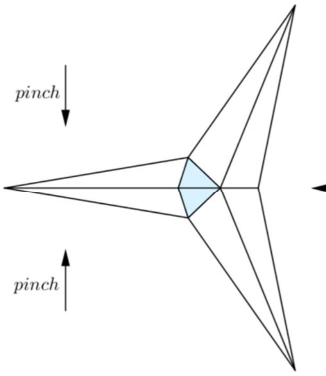
Step 5



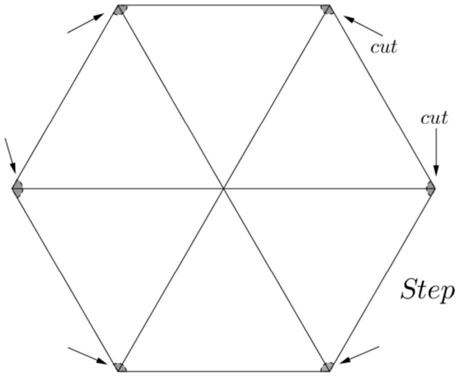
Step 6

6. Final step is to fold triangle 20 on triangle 11. You get a regular hexagon as in the picture. If you glue triangle 11 and 20 together, the flexagon is ready.

7. How to flex? Choose two adjacent triangles and pinch them together. Push the middle of the other 4 triangles towards the center (you get a 'windmill' like in the picture below). If you choose two good triangles to pinch, the center will open a little and the triangles that were hidden will appear.
   If the center does not open: you'll have to pinch another couple of triangles (it is wise to take one of the triangles you tried first, but pinch it together with his other adjacent triangle).



Step 7



Step 8

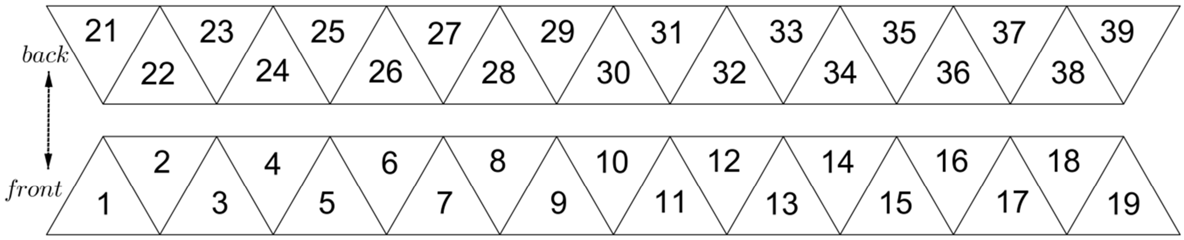
8. When flexing is not going very smoothly, you probably did not fold precisely enough. One way to solve this, is to cut in every edge a little bit as in the picture. Then there will be more space to flex.

Assignment 4

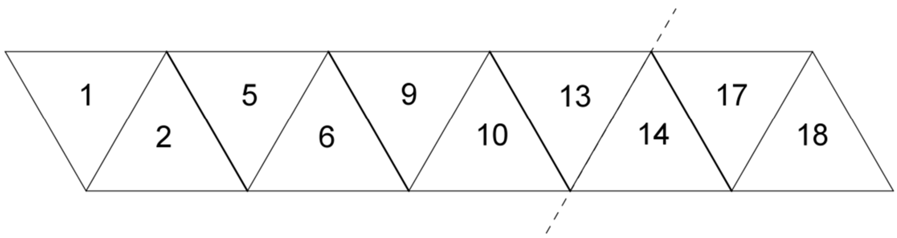
- ✓ Start with an arbitrary folding line on your paper roll.
- ✓ Do action 'Folding down' (D) to create a new folding line
- ✓ Do action 'Folding Up' (U) to create e new folding line
- ✓ Repeat this (doing D, then U) until you have 23 'triangles' on your paper strip
- ✓ Can you see that the triangle seems to become more equilateral as we proceed folding? Now, cut of the strip with folding lines. This gives you a strip of 23 triangles.
- ✓ Cut off the first 4 triangles (make sure that you cut off the triangles that were folded first). So you are left with a strip of exactly 19 triangles, which are all more or less equilateral.
- ✓ Make a flexagon according to instructions and learn to flex.
- ✓ Color each face you see in a different color.
- ✓ Make a scheme where you give every possibility of your flexagon (front/back color), and how you can get from one to another.
- ✓ Which two colors are separated by the longest distance, when we define 'distance' as 'number of flexes you have to do in order to get from one color to another'.

Folding instructions

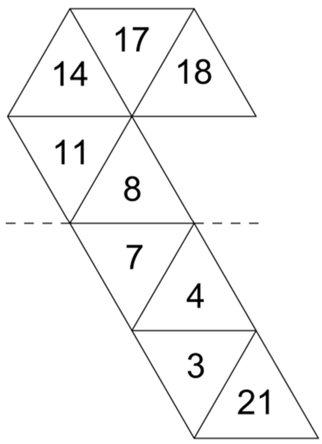
1. Number the 19 triangles as in the figure: make sure that triangle 1 has its top upwards. We do not have a triangle 20: the back of a triangle is 20 more than the front number (so the back of triangle 1 is triangle 21,…). It is easier if you color already the triangles softly. Triangles 1,4,7,10,13,16 have color 1 (you may choose);; 2,5,8,11,14,17 have color 2;; 3,6,9,12,15,18 have color 3;; 22,23,28,29,34,35 have color 4;; 24,25,30,31,36,37 have color 5;; 26,27,32,33,38,39 have color 6. Remark that triangles 19 and 21 remains without color.
Make sure all folding lines are folded in both directions. We need all lines to be very flexible!!

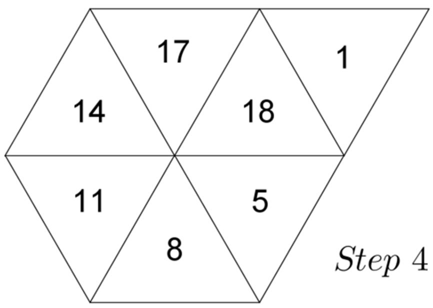


2. Start with the front. Start to fold on the line between triangle 2 and 3: put triangles 22 and 23 on top of each other (= folding underneath, as in remark below left picture). Fold further until triangle 5 lies next to triangle 2 by placing triangles 24 and 25 on top of each other (see remark below, right picture).
Do analogous actions starting with putting triangles 26 and 27 on top of each other,…
You make from the strip a kind of flat spiral that looks, after rotating, in the front like the picture below.



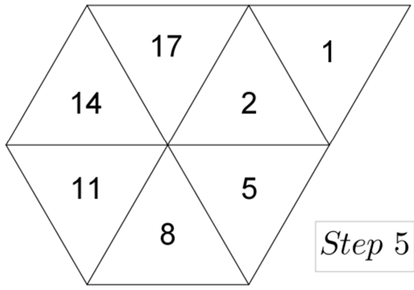
3. Now fold the left part underneath the right part around the edge between triangle 13 and 14. You get something like the picture below left.



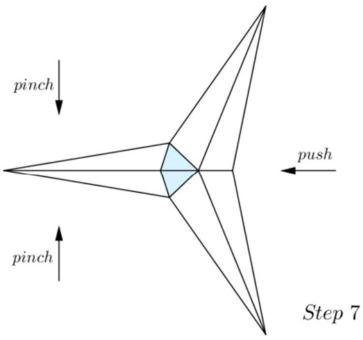


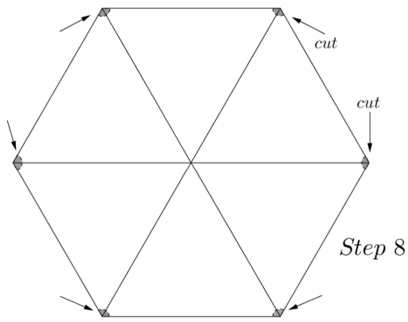
4. Then fold underneath along edge 8-7 (see picture left above) and you get picture above right.

5. The last part of the strip should now be placed on top of triangle 18 (or otherwise, place triangle 18 underneath the last part of the strip). You get the picture as on the right.



6. Now fold triangle 1 to the back. If you folded everything correctly, you'll see that by this action triangle 21 comes on top of triangle 19. Glue these two triangles to each other. Congrats! You now have a flexagon with 6 different colors.
7. Now we learn to 'flex'. Choose two adjacent triangles and pinch them together. Push the middle of the other 4 triangles towards the center (you get a 'windmill' like in the picture below). If you choose two good triangles to pinch, the center will open a little and the triangles that were hidden will appear.
   If the center does not open: you'll have to pinch another couple of triangles (it is wise to take one of the triangles you tried first, but pinch it together with his other adjacent triangle).

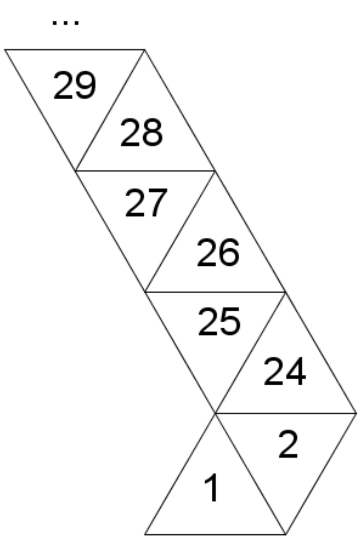


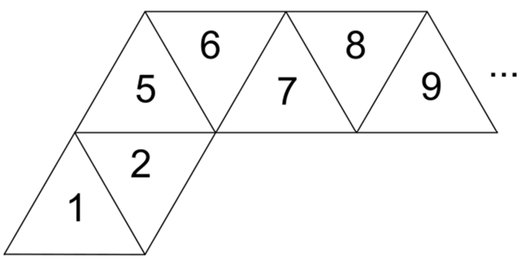


8. When flexing is not going very smoothly, you probably did not fold precisely enough. One way to solve this, is to cut in every edge a little bit as in the picture. Then there will be more space to flex.

<u>!! Remark (from step 1 to step 2) !!</u>

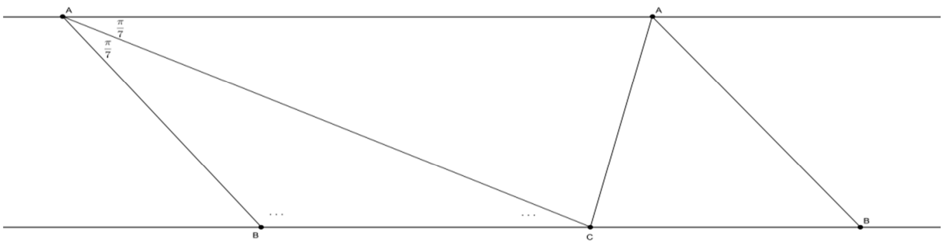
In the original assignment presented on the KUPM 2014, the triangle numbers in step 2 were incorrect. These are corrected now, and I also added two additional pictures that are intermediate steps between the pictures of step 1 and step 2.





Assignment 5

- ✓ First we want to understand why a D_2U_1-strip can be folded into a regular heptagon (as we did in the introduction). It all comes down to 'angles'. So calculate the exterior angles in a regular heptagon.
- ✓ Now look at the D_2U_1-strip and suppose that you started with angle $2\pi/7$ between AB and the above edge of the strip. You can see that folding line AC is bisecting this angle and therefore we get two angles of $\pi/7$ (see picture below). Do calculate the other angles in the D_2U_1-strip. What do you notice?



- ✓ Explain why $2\pi/7$ is indeed the right angle to look for when we do the folding as in the introduction?
- ✓ Suppose that you did not start with the right angle $2\pi/7$, but with another angle. We call this start angle $2\pi/7 + e$. The letter e stands for 'error' and can be both positive and negative. Calculate now the (first four) angles of the D_2U_1-strip and describe what you see. Can you conclude that you are able to fold a regular heptagon without using a protractor.

- ✓ The question that we want to discuss now is the following: if you have a regular polygon, can you calculate the way to fold the corresponding strip? The answer is rather easy when your polygon has an odd number of edges, say n. We can define a folding symbol as follows:

1. all a_i are odd
2. $a_1 = 1$
3. $n - a_i = 2^{b_i}.a_{i+1}$
4. $a_{m+1} = a_1 = 1$

n					
	a_1	a_2	a_3	...	a_m
	b_1	b_2	b_3	...	b_m

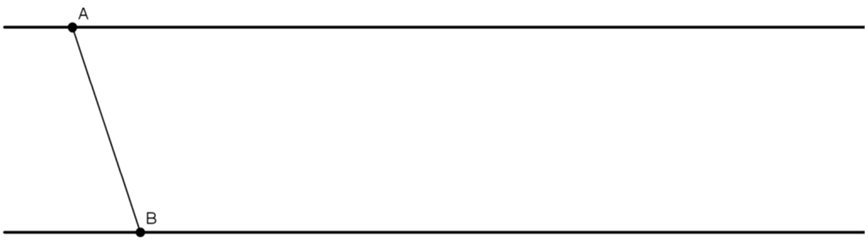
If you calculate the symbol for the heptagon, you get (check this) :

7		
	1	3
	1	2

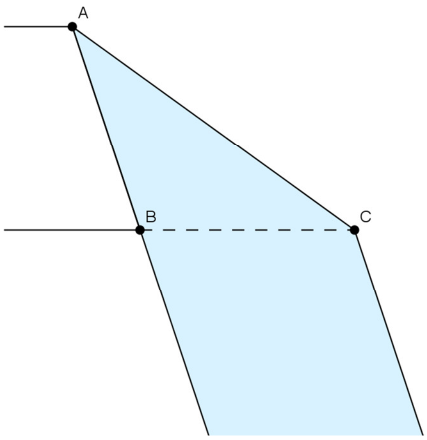
- ✓ We explain the symbol. In the second row we can read off how we should make the corresponding strip: U_1D_2 (or D_1U_2) is a corresponding strip to work with in order to get (the right angle of) a regular heptagon. The numbers of the first row, tell you about which regular polygons are possible with this strip (see proceedings for more explanation).
- ✓ Suppose you want to fold a regular pentagon. Calculate the corresponding folding symbol.
- ✓ What is the corresponding strip looks like? Make this folding strip and make a regular pentagon.
- ✓ Try to explain what you did by discussing the angles.

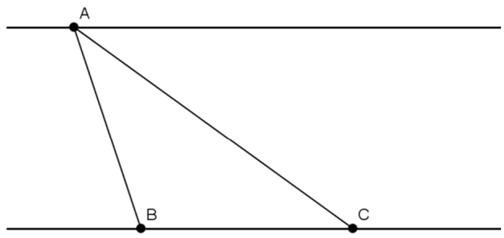
- Extra. Calculate the folding symbol if you want to fold a regular 31-gon. Can you say how you can fold the corresponding strip?

Folding Actions (create a new folding line starting from a given one)



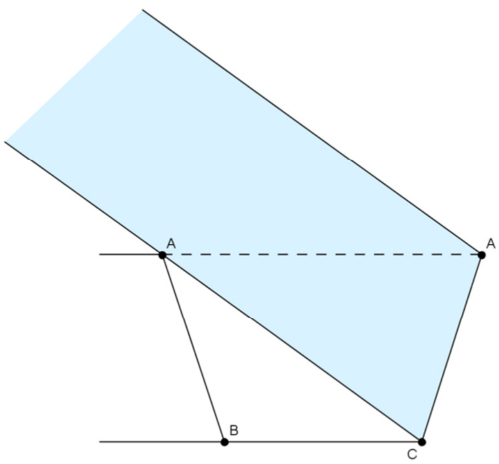
First, you can fold downwards (we will call this folding D) starting from line AB. You put the upper edge of the paper roll from point A on top of the fold line AB (see left-hand drawing, where the back of the strip is colored). This creates a new folding line in a natural way. If you fold your paper roll back open, you have a new line AC visible on the paper roll as in the right-hand picture.





Folding Downwards (D)

You can fold upwards (we will call this folding U): we start from the line AC. You then put the bottom edge from point C along the line AC (see picture below on the left). You get another new folding line. When you unfold the paper roll you have a new visible line CA.



Folding Upwards (U)