Paper roll mathematics in the classroom

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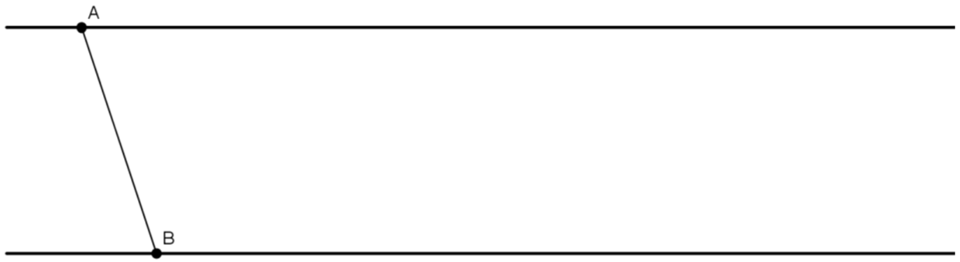
Folding with a paper roll?

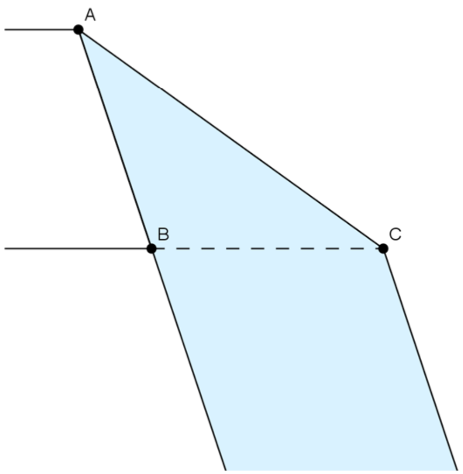
- Making folding lines on a very long strip of paper (= the paper roll)
- Start with an arbitrary line
- Only the last line is needed to make a new folding line! We make folding lines from left to right!

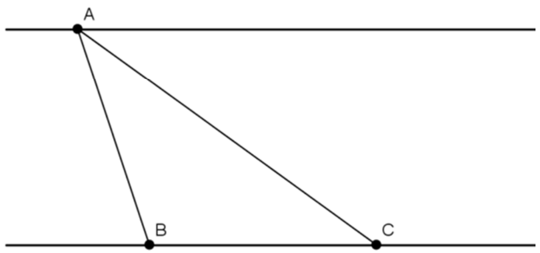
- There are two sort of foldings: folding Downwards (D) and folding Upwards (U)

D: folding downwards

Fold the top edge of the paper roll on top of the last line, by folding (the right part of) the paper roll downwards: you get a new folding line.



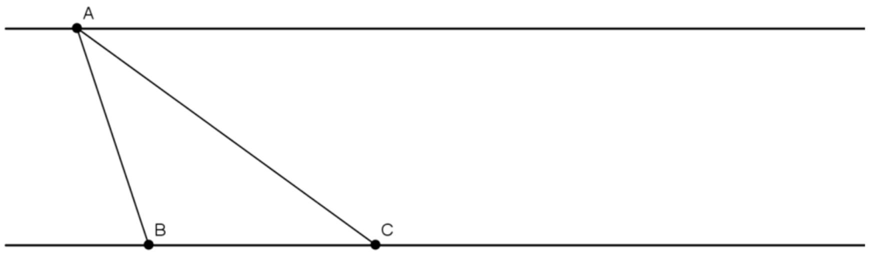


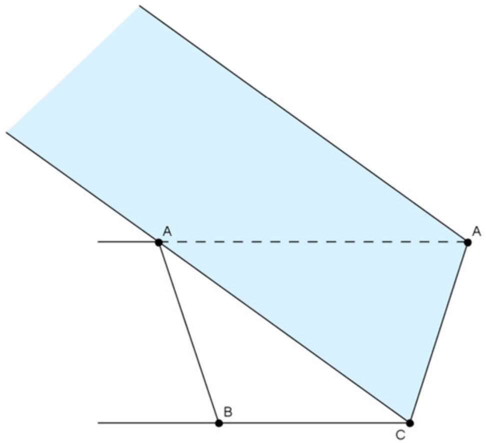


Folding Downwards (D)

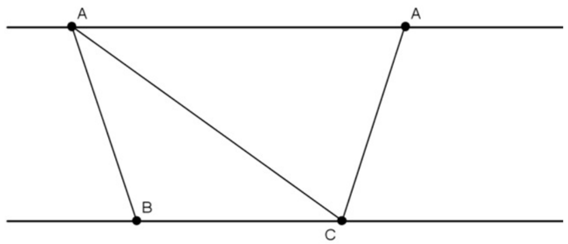
U: folding upwards

Fold the bottom edge of the paper roll on top of the last line, by folding (the right part of) the paper roll upwards: you get a new folding line.





Folding Upwards (U)



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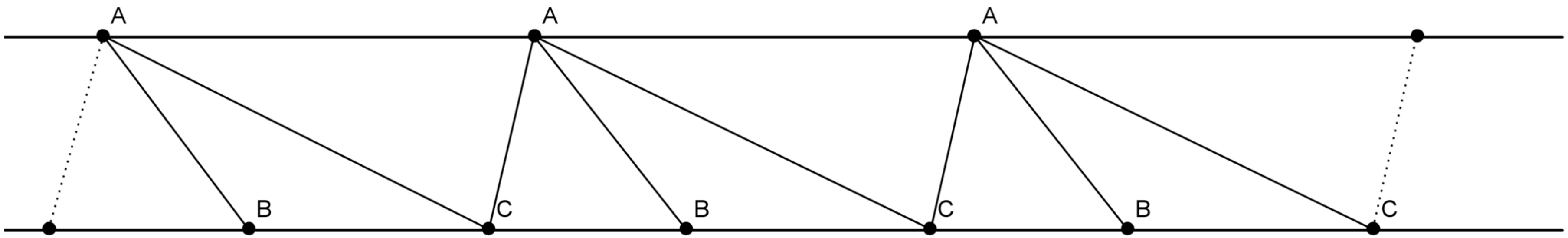
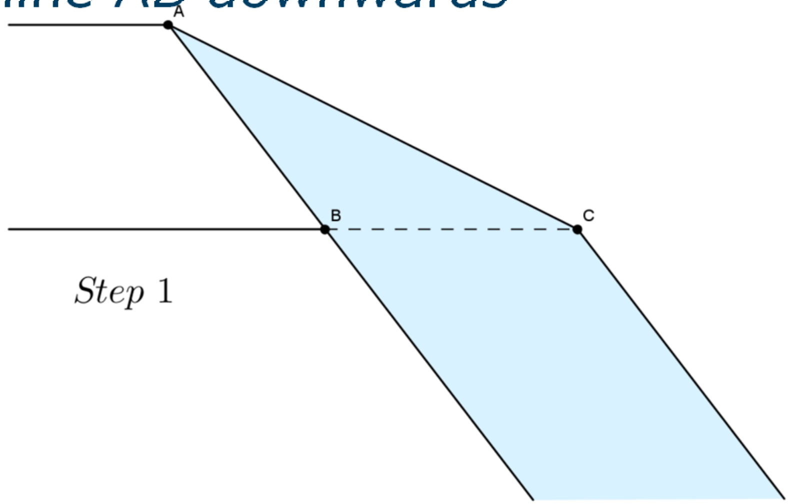
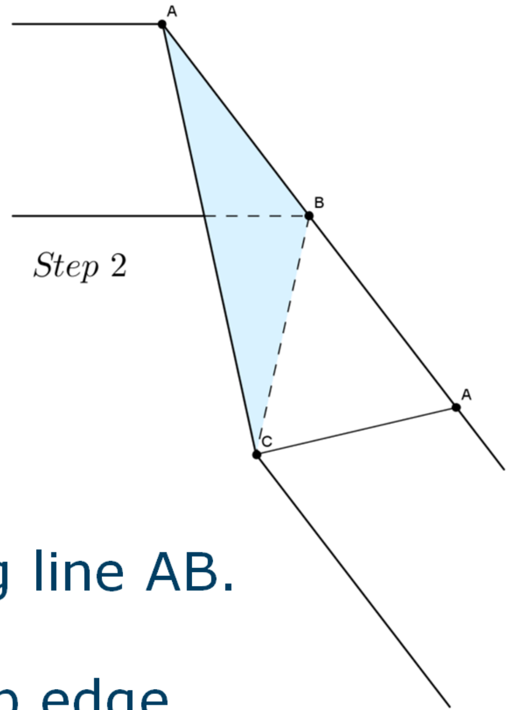


- The strip in front of you, is made by repeating D_2U_1, i.e. two times folding downwards followed by one time folding upwards.
- Wich ‘figures’ can be folded with this strip? Can you see symmetry?

Folding regular polygons

1. Fold the top edge of the paper roll on the (most) left line AB downwards

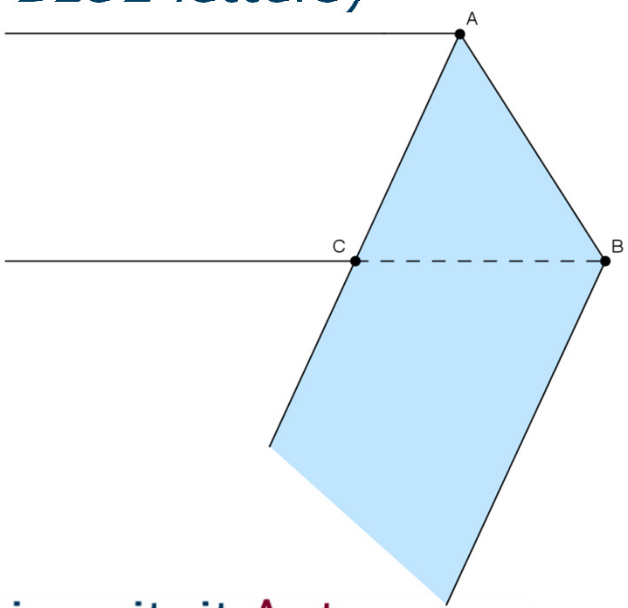


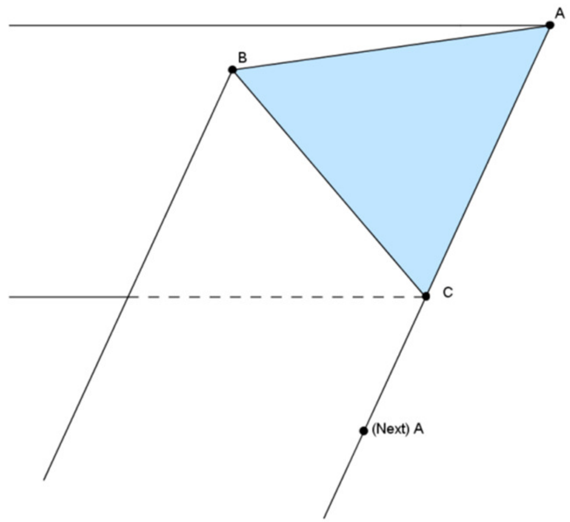


2. Turn the strip further by folding along line AB.
   (Sort of spiral movement)
   Point A will be again visible on the top edge.
3. Keep repeating this, starting with the next A on your strip (Strip with RED letters)

Another one

1. Fold the top edge of the paper roll on the (most) left line AC downwards
2. Turn the strip further by folding along line AC (spiral movement)
3. Keep repeating this, starting at the next A (strips with BLUE letters)





Result?

Look at the outside border of the folded figure. What do you notice?

Figure 1 (red letters)

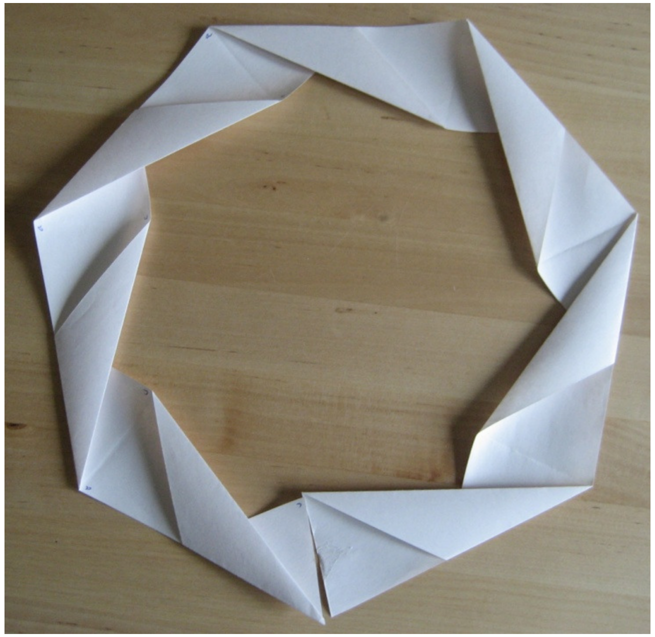
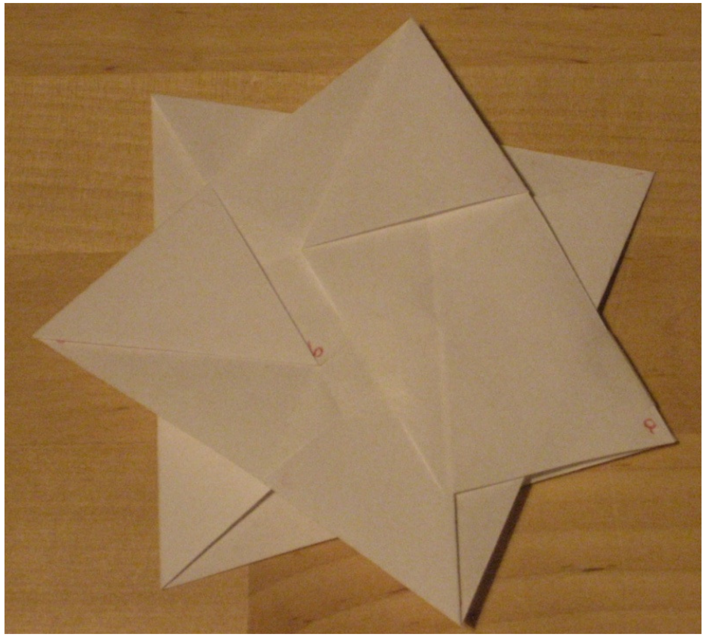
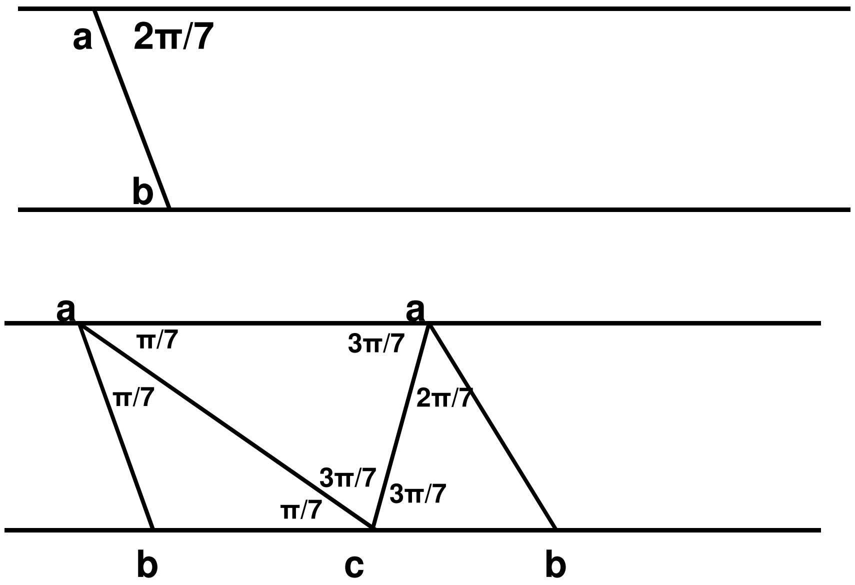


Figure 2 (blue letters)



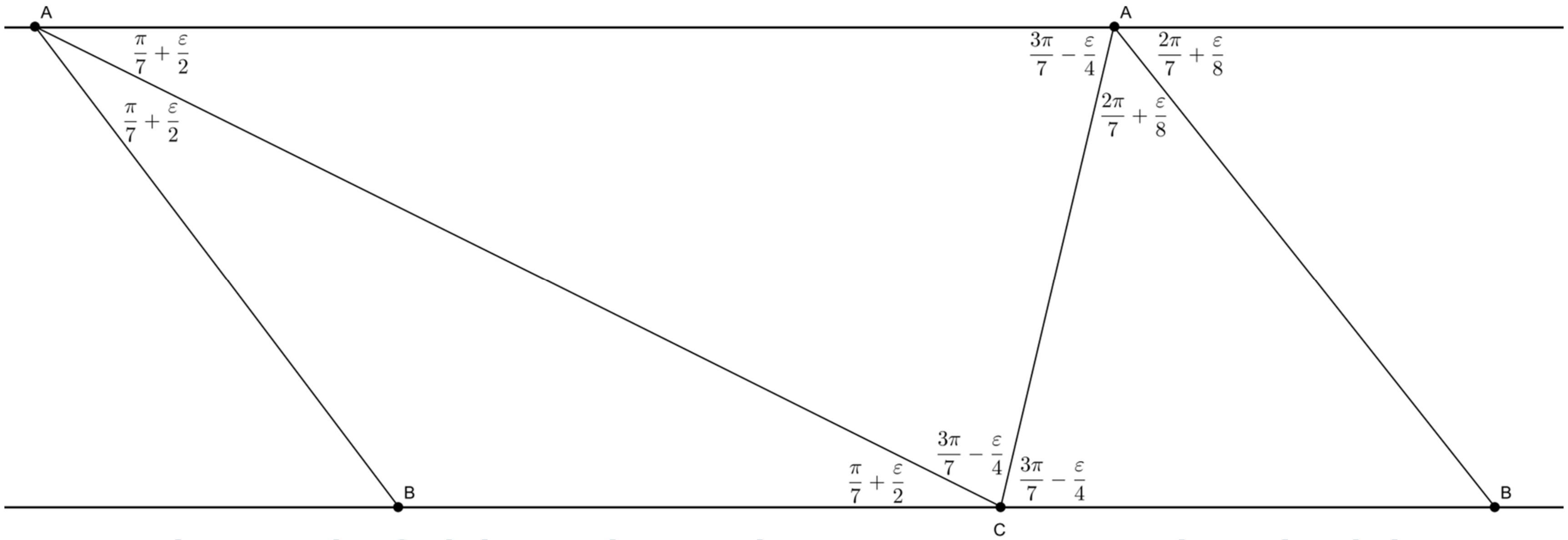
Why a heptagon? (Assign. 5)

- Calculate the exterior angle in a regular heptagon: this is equal to 2π/7.
- Suppose, by accident, that your first line made an angle of 2π/7 with the top edge….



Heptagon: convergence

- Suppose you start with another angle: 2π/7 + e (e can be positive/negative)



- With each folding line the error gets divided by 2 → exponential convergence

Folding symbol

- The calculation rules:

1. all a_i are odd
2. $a_1=1$
3. $n-a_i = 2^{b_i}.a_{i+1}$
4. $a_{m+1}=a_1=1$

$$\mathbf{n}\left|\begin{array}{ccccc} a_1 & a_2 & a_3 & \dots & a_m \\ b_1 & b_2 & b_3 & \dots & b_m \end{array}\right|$$

- Example: heptagon

  $7-1 = 6 = 2^1.3$

  $7-3 = 4 = 2^2.1$

$$\mathbf{7}\left|\begin{array}{cc} 1 & 3 \\ 1 & 2 \end{array}\right|$$

- Second row: tells you how you should fold the paper strip

Little Exercise (Assign. 5)

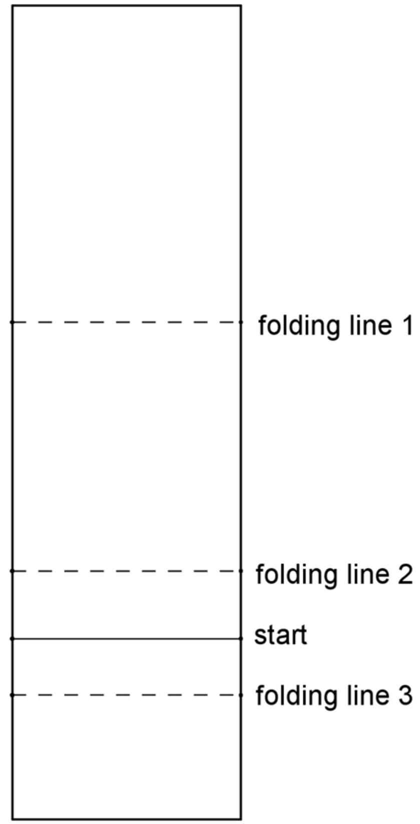
- Calculate the symbol for regular 31-gon

31	1	15
	1	4

- Conclusion: folding D_4U_1 (or U_4D_1) gives a strip with the required angle to fold a regular 31-gon

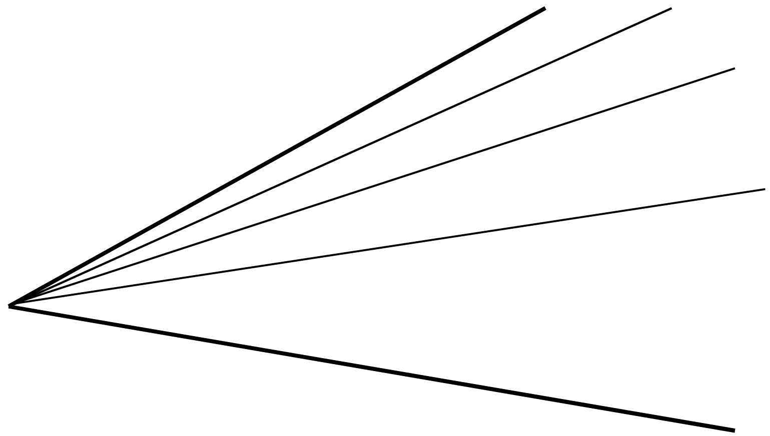
Dividing a rectangle (Assign. 3)

- Procedure D_1U_2 can be used to divide a rectangle in 7 equal pieces.
- Starting with an arbitrary guess (heigth 1/7+e).
- Line 1 will be at height 4/7 + e/2
  Line 2 will be at height 2/7 + e/4
  Line 3 will be at height 1/7 + e/8
- Also 2/7th, 3/7th,… possible
- Procedure D_2U_2 can be used to fold into a/5th piece for every a.



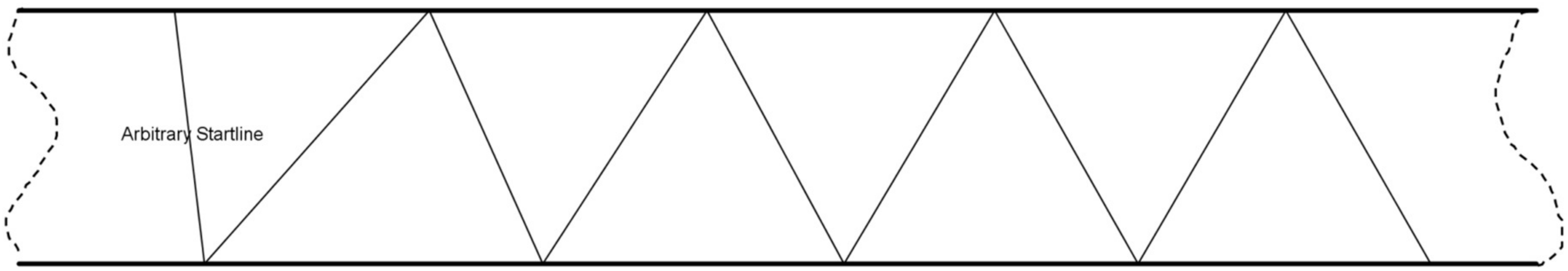
Dividing angles...

- In fact: the folding procedure gives a way to 'converge' to a rational number a/n.
- You can fold 1/3th of an arbitrary angle by (repeating) procedure U_1D_1.



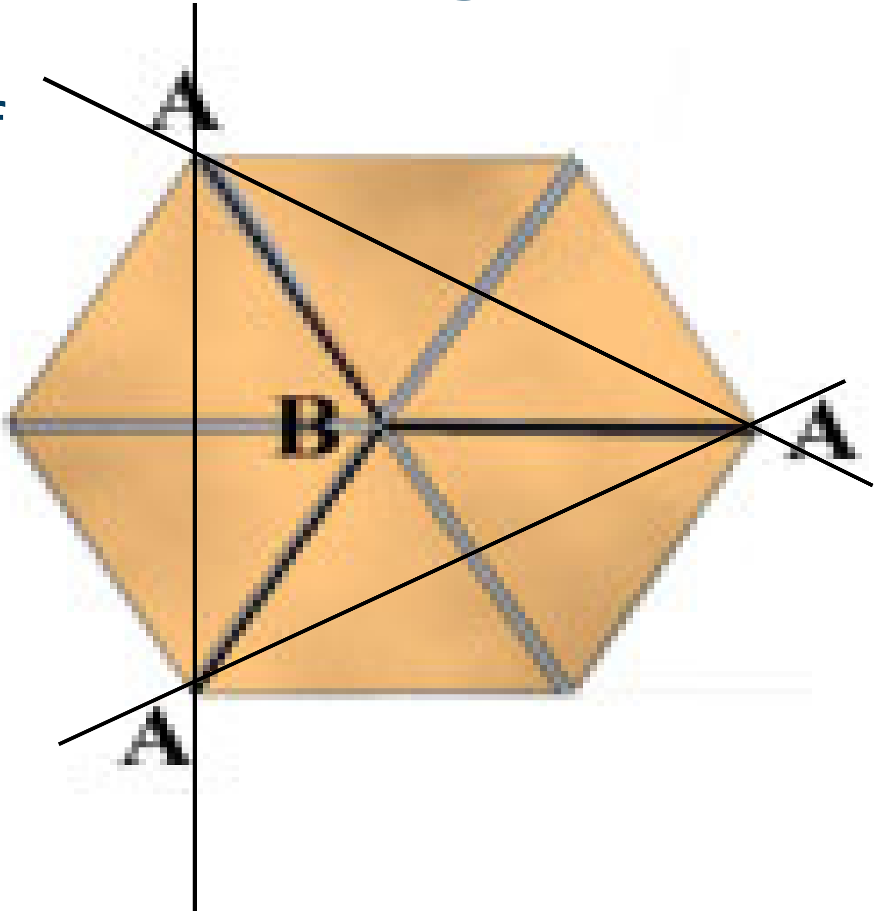
Flexagons

- "Flexible" polygons (Artur Stone 1939)
- You need strip of 10 equilateral triangles for flexagon with 3 faces (Assign. 1 & 2)
- This can be done by repeating folding D_1U_1 (without protractor). You have again exponential convergence.



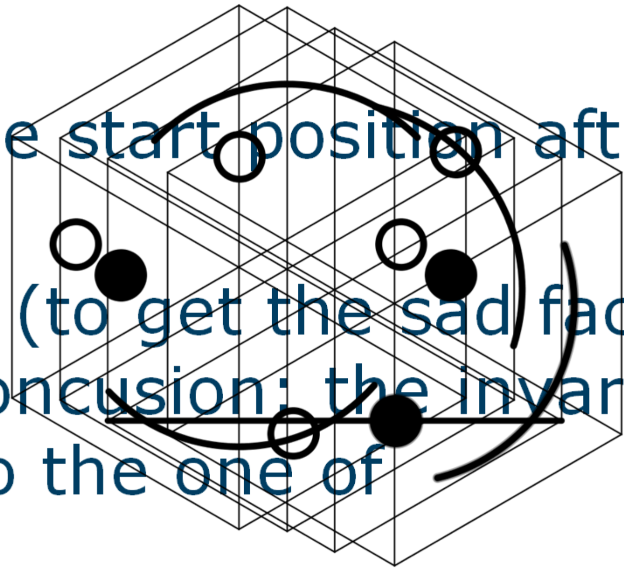
Flexing movement (Assign. 1)

- By flexing, every "pair of triangles" gets rotated around a central axis.
- Since these axis have an angle of 60°, the corresponding "pairs of triangles" make a rotation of 120° with respect to each other.
- 3D geometry gets full of surprises (happy/sad face)

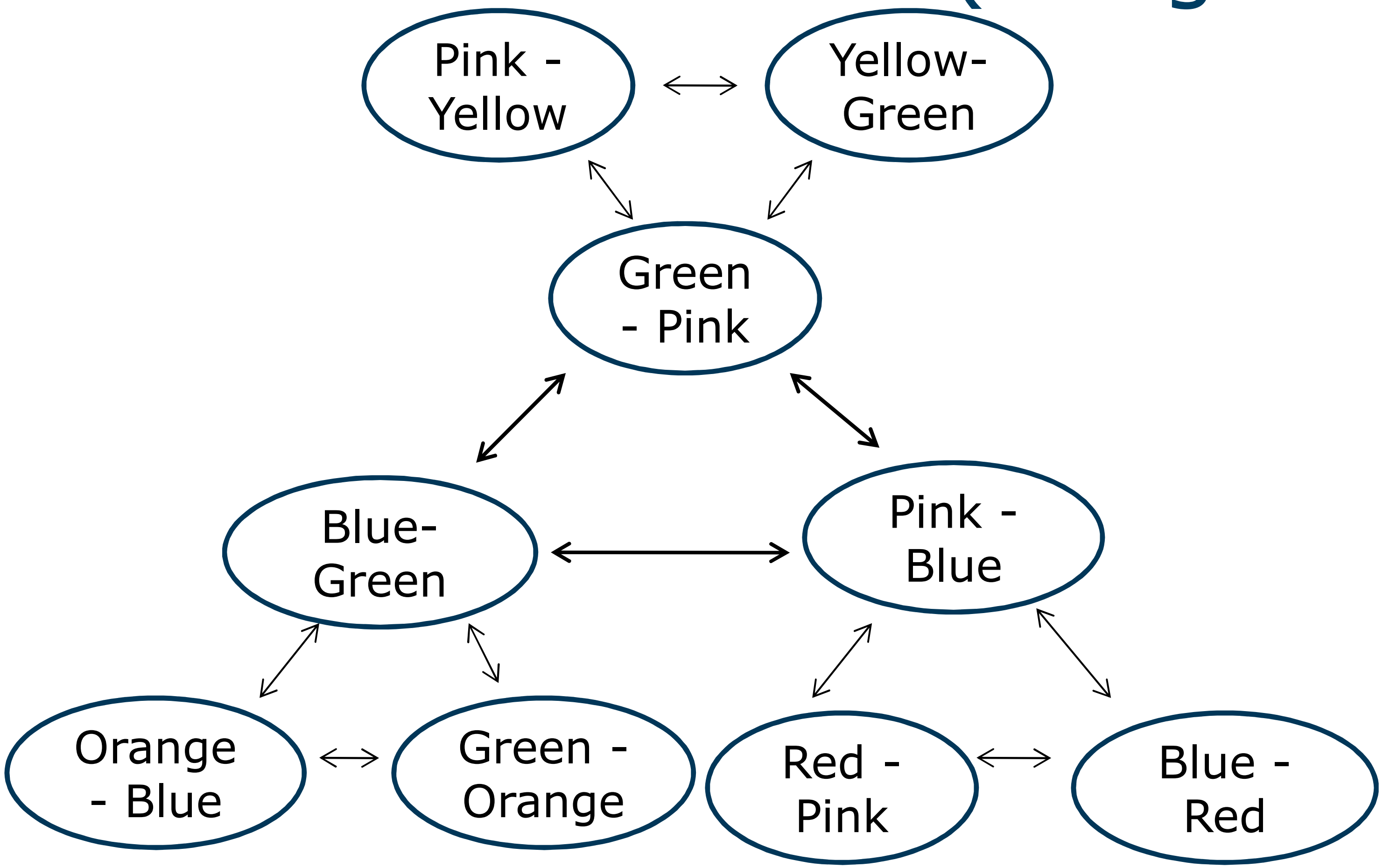


Invariance group (Assign. 2)

- Since there are only three faces, you expect that after 3 flexes, you are back to the beginning position.
- If you look very carefully: after 3 flexes, you end up with the same face, but it has rotated 60°
- So, you only get to the start position after 18 flexes.
- Together with the flip (to get the sad faces), we end up with the concusion: the invariance group is isomorphic to the one of a regular 18-gon.



Scheme of possibilities (Assign. 4)



Remarks

- Pupils learn to work very carefully and with great precision: e.g. folding procedure
- There are flexagons with 3, 4, 5, 6, 7 different faces.
- Also other figures besides triangles are possible
- Possibility to add a personal touch to mathematics with drawings, photos

Didactic conclusions

- Mathematics as an explanation of visible phenomena, explaining what one experiences
- Variation in topics
- Variation in difficulty, level of assignments
- Opportunities of surprising and challenging your pupils
- Opportunities to add a personal touch
- (Interdisciplinary) curricular opportunities
- Simple and cheap material
- Opportunities for further self-discovery

This evening at 17h (in hotel Toplice):
An international initiative to stimulate research competences in mathematics
→ For teachers for pupils 17+

Questions? Remarks?

References

1. Hilton P., Holton D., Pedersen J., Mathematical Reflections: In a Room with Many Mirrors, Springer-Verlag, New York 1998, 351p.
2. Hilton P., Holton D., Pedersen J., Mathematical Vistas: From a Room with Many Windows, Springer-Verlag, New York 2002, 358p.
3. Hilton P., Pedersen J., Walser H., The faces of the tri-hexaflexagon, Mathematics Magazine vol. 70, October 1997, p.243-251
4. Polster B., Variations on a Theme in Paper Folding, American Mathematical Monthly nr. 111, January 2004, p.39-47
5. Pook L., Flexagons Inside Out, Cambridge University Press 2003, 182p.
6. Tekulve A., Understanding Polygons and Polyhedrons Using Flexagons, The Montana Mathematics Enthusiast, vol. 1, nr. 1, 2004, p. 20-28.